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LETTER TO THE EDITOR

Integrable Hamiltonians with higher transcendental invariants

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Abstract. A new family of integrable two-dimensional Hamiltonians is found. The corresponding potentials $V(x, y)$ are determined by the relation $\Phi(V)-y-x/(a_1V+a_2)=0$, where Φ is an arbitrary function and a_1, a_2 are arbitrary parameters. The second constants of motion are higher transcendental functions in momenta. These results are found by deriving non-separable solutions of the Hamilton–Jacobi equation.

Finite-dimensional deterministic dynamical systems have been the subject of intensive research in recent years. Conservative, as well as dissipative, systems have been extensively studied fundamentally with respect to their erratic or chaotic behaviour.

The typical representatives of conservative systems are Hamiltonian problems of classical mechanics which are of great interest to many branches of science like celestial mechanics, plasma physics, accelerator dynamics, etc [1, 2]. A Hamiltonian system is said to be completely integrable in the sense of Liouville if it is possible to find N independent constants of motion, where N is the number of degrees of freedom (also called the dimension) of the system. As is well known, the question of integrability of Hamiltonian problems is one of the oldest problems in this field.

Hamiltonian systems with two degrees of freedom are the simplest problems of classical mechanics with non-trivial behaviour: their equations of motion are, in general, nonlinear and coupled in a way that makes them non-solvable by standard mathematical techniques. In fact, in most cases, they are not Liouville integrable and have large scale regions in their phase space within which the orbits wander in a chaotic fashion [1-3]. Nevertheless, a great number of two-dimensional integrable Hamiltonians have been found in recent years [4, 5]. In general, for a given Hamiltonian, there does not exist a necessary and sufficient integrability criterion. The so-called 'Painlevé property' (for reviews see [5] and [6] and references therein) seems to be a sufficient but not necessary condition for integrability. On the other hand, the non-integrability theorems of Ziglin [7] and Yoshida [8] are not yet technically applicable for arbitrary potentials.

A two-dimensional time-independent Hamiltonian will be integrable if it is possible to find a second constant of motion (the first is the Hamiltonian). To obtain it one has to solve the partial differential equation resulting from the requirement that the Poisson bracket between the Hamiltonian and the second invariant vanishes. However, there is no general method to solve this equation and, therefore, some simplifying ansätze are derived [4]. In general, it has almost always been assumed that the second invariant is polynomial in momenta. In this way, a great number of integrable models

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have been found [4]. Also, an example of an integrable Hamiltonian with a second invariant given by a *higher* transcendental function in momenta has been presented in [9]. This case corresponds to the potential $V = x/y$.

In this letter a new family of integrable two-dimensional Hamiltonians is found, which contains as a particular case the potential $V = x/y$ mentioned above. For all potentials of this family the second constant of motion is a higher transcendental function in momenta. These results are deduced by finding non-separable solutions of the Hamilton–Jacobi (HJ) equation, as follows. Let us consider the Hamiltonian:

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(x, y). \tag{1}$$

The HJ equation for the characteristic function $S(x, y)$ is [10]:

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial y}\right)^2 + V = E \tag{2}$$

where E is the constant energy of the system.

In general, the solutions that have been found for the HJ equation, correspond to 'separable' potentials (i.e. potentials for which the HJ equation becomes separable after an adequate change of coordinates) [10]. In this work we present a procedure that enables us to find solutions of the HJ equation for a certain class of non-separable potentials. The essential point of this procedure consists of proposing a solution of (2) of the following form:

$$\frac{\partial S}{\partial x} = F_1(V) \qquad \frac{\partial S}{\partial y} = F_2(V) \tag{3}$$

where F_1 and F_2 are functions only of V. By using (2), F_2 is determined in terms of F_1:

$$F_2(V) = \sqrt{2(E - V) - F_1^2(V)}. \tag{4}$$

Obviously, the compatibility condition $\partial^2 S/\partial x \partial y = \partial^2 S/\partial y \partial x$ must also be imposed, from which we obtain

$$\frac{\mathrm{d}F_1}{\mathrm{d}V}\frac{\partial V}{\partial y} = \frac{\mathrm{d}F_2}{\mathrm{d}V}\frac{\partial V}{\partial x}. \tag{5}$$

Taking into account (4), equation (5) becomes

$$\frac{\mathrm{d}F_1}{\mathrm{d}V}\frac{\partial V}{\partial y} + \frac{1 + F_1\,\mathrm{d}F_1/\mathrm{d}V}{\sqrt{2(E - V) - F_1^2}}\frac{\partial V}{\partial x} = 0. \tag{6}$$

As F_1 is a function only of V, in order to ensure the coherence of (6), the following condition must be imposed on the potential

$$\frac{\partial V}{\partial y} = f(V)\frac{\partial V}{\partial x} \tag{7}$$

where f is an arbitrary function of V. For a given function f, it is easy to show, by using the method of characteristic [11], that the general solution of the partial differential equation (7) for the potential is:

$$\Phi(V) - y - \frac{x}{f(V)} = 0 \tag{8}$$

where Φ is an arbitrary function of V.

When (7) is substituted in (6), we obtain an ordinary differential equation for F_1, given by:

$$\frac{dF_1}{dV} f(V) + \frac{1 + F_1\, dF_1/dV}{\sqrt{2(E-V) - F_1^2}} = 0. \tag{9}$$

For a fixed function f, there is a family of potentials defined in an implicit way by (8). For each function Φ, a member of this family is obtained. The HJ equation for the corresponding potential is reduced to the first-order ordinary differential equation (9). This equation can be simplified by introducing a new function M as follows:

$$M^2 = 2(E-V) - F_1^2. \tag{10}$$

It is not difficult to show that, using (10), equation (9) becomes

$$\frac{dM}{dF_1} = f(-\tfrac{1}{2}(M^2 + F_1^2) + E) \tag{11}$$

where we have considered F_1 as the independent variable. The only integrable case of (11) corresponds, according to the Painlevé analysis for first order ordinary differential equations [11], to a linear function f:

$$f(V) = a_1 V + a_2 \tag{12}$$

where a_1 and a_2 are arbitrary parameters.

For this case, equation (11) becomes

$$\frac{dM}{dF_1} + \frac{a_1}{2} M^2 = -\frac{a_1}{2} F_1^2 + a_1 E + a_2 \tag{13}$$

which is particular case of the Riccati equation [11]. Its general solution is given by

$$M(F_1) = \frac{2}{\sqrt{a_1}} \frac{K(dW/du)(\alpha, u) + (dW/du)(\alpha, u)}{KW_+(\alpha, u) + W_-(\alpha, u)} \tag{14}$$

where K is the constant of integration, W_+ and W_- are the two standard parabolic cylinder functions [12], i.e. the two independent solutions of the equation $v''(u) + (u^2/4 - \alpha)v(u) = 0$. The quantities α and u are given by

$$\begin{aligned} \alpha &= \tfrac{1}{2}(a_1 E + a_2) \\ u &= \sqrt{a_1}\, F_1. \end{aligned} \tag{15}$$

From (10) and (14) and the relation $\partial S/\partial x = p_x$ we obtain:

$$\tfrac{1}{2}a_1(E-V) - \tfrac{1}{4}a_1 p_x^2 = \left(\frac{KW'_+ + W'_-}{KW_+ + W_-}\right)^2 \tag{16}$$

where the primes indicate a derivative with respect to u. From this relation, the constant K can be expressed in terms of p_x, p_y and E:

$$K = \frac{\sqrt{a_1} p_y W_-(\frac{1}{2}a_1 E + a_2, \sqrt{a_1} p_x) - 2W'_-(\frac{1}{2}a_1 E + a_2, \sqrt{a_1} p_x)}{2W'_+(\frac{1}{2}a_1 E + a_2, \sqrt{a_1} p_x) + p_y W_+(\frac{1}{2}a_1 E + a_2, \sqrt{a_1} p_x)}. \tag{17}$$

This expression gives the second constant of motion for the family of potentials determined by (8) and (12). Since the parabolic cylinder functions are entire analytic functions for all values of the parameters, we see that the second constant of motion is a single-valued meromorphic function as required by the Liouville theorem. Therefore, we have shown that the potentials determined by (8) and (12) are integrable, by explicitly determining the second constant of motion (17). We have obtained this result by finding non-separable solutions of the HJ equation, through the ansatz (3).

If we take $\Phi=0$, $a_1=-1$ and $a_2=0$ in (8) and (12), we obtain the potential $V=x/y$ found by Hietarinta in [9], with the corresponding constant of motion deduced from (17) by taking $a_1=-1$ and $a_2=0$. For other functions Φ the resulting potentials are not rational functions of x and y. Algebraic or transcendental expressions are obtained, such as for example the potential

$$V=\tfrac{1}{2}(a_1y+\sqrt{a_1^2y^2+4a_2y+4x}) \tag{18}$$

It is interesting to note that if the energy E is replaced by the Hamiltonian in (17), the constant of motion develops, as a function of the coordinates x and y, more complicated singularities than a simple pole. In fact, expression (17) is a 'simple' meromorphic function only on each surface of constant energy E. It is not a global meromorphic function in the complete phase space.

As it has been shown by Hietarinta in [9], the 'weak Painlevé' analysis does not work for this type of potential (more precisely for the particular case $V=x/y$, but it is reasonable to assume that his analysis is valid for all potentials defined by (8) and (12)). Therefore, the coordinates x and y have, as functions of time, more complicated movable singularities than poles.

This fact enables us to understand why the second constant of motion (17) is not globally analytical in phase space. Its 'additional' singularities are necessary in order to compensate the more complex structure of singularities (when compared with cases for which the 'weak Painlevé' property is satisfied) of x and y as functions of time. This compensation is necessary in order to obtain a constant of motion after replacing all phase space variables in (17) as functions of time.

Finally, it would be interesting to generalise the method applied in this work to higher-dimensional Hamiltonians.

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